106 中正

a) (5%) Find an LU-Decomposition of the following matrix $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$

b) (5%) Use the solution of part(a) to solve the following system of linear equations

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 &= 2 \\ -3x_1 - 8x_2 &= 2 \\ 4x_1 + 9x_2 + 2x_3 &= 3 \end{aligned}$$

(No credit for other methods)

(a)

$$A = \begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix} \xrightarrow{r_{12}^{(1.5)},\, r_{13}^{(-2)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow{r_{23}^{(3)}} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix} = U$$

故 $R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)} A = U$, 即 $A = [R_{23}^{(3)} R_{13}^{(-2)} R_{12}^{(1.5)}]^{-1} U = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} U$,

取 $L = R_{12}^{(-1.5)} R_{13}^{(2)} R_{23}^{(-3)} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$,

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得 $A = LU = \begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}$.

(b)

故 $Ax = b \Leftrightarrow LUx = b$,

令 $Ux = y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, 先解 $Ly = b$: $\begin{bmatrix} 1 & 0 & 0 \\ -1.5 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$, 得 $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}$,

再由 $Ux = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, 即 $\begin{bmatrix} 2 & 6 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 14 \end{bmatrix}$, 解得 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

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6. (10%) Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$. Find the QR-decomposition of A.

令 $\boldsymbol{v}_1=\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \boldsymbol{v}_2=\begin{bmatrix}-1\\2\\1\\0\end{bmatrix}, \boldsymbol{v}_3=\begin{bmatrix}1\\1\\2\\0\end{bmatrix}$,用 $Gram-Schmidt\ process$ 對 $\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3$ 作正交化.

$$\boldsymbol{u}_1=\boldsymbol{v}_1=\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, <\boldsymbol{u}_1,\boldsymbol{u}_1>=2,$$

$$\boldsymbol{u}_2=\boldsymbol{v}_2-\frac{<\boldsymbol{v}_2,\boldsymbol{u}_1>}{<\boldsymbol{u}_1,\boldsymbol{u}_1>}\boldsymbol{u}_1=\begin{bmatrix}-1\\2\\1\\0\end{bmatrix}-\frac{-1}{2}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}=\begin{bmatrix}\frac{-1}{2}\\2\\1\\\frac{1}{2}\end{bmatrix},\ <\boldsymbol{u}_2,\boldsymbol{u}_2>=\frac{11}{2}.$$

$$\boldsymbol{u}_3=\boldsymbol{v}_3-\frac{<\boldsymbol{v}_3,\boldsymbol{u}_1>}{<\boldsymbol{u}_1,\boldsymbol{u}_1>}\boldsymbol{u}_1-\frac{<\boldsymbol{v}_3,\boldsymbol{u}_2>}{<\boldsymbol{u}_2,\boldsymbol{u}_2>}\boldsymbol{u}_2=\begin{bmatrix}1\\1\\2\\0\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}-\frac{7/2}{11/2}\begin{bmatrix}\frac{-1}{2}\\2\\1\\\frac{1}{2}\end{bmatrix}=\frac{1}{11}\begin{bmatrix}9\\-3\\15\\-9\end{bmatrix}, <\boldsymbol{u}_3,\boldsymbol{u}_3>=\frac{396}{121}$$

$$\therefore\begin{cases}\boldsymbol{v}_1=\boldsymbol{u}_1\\ \boldsymbol{v}_2=\frac{-1}{2}\boldsymbol{u}_1+\boldsymbol{u}_2\\ \boldsymbol{v}_3=\frac{1}{2}\boldsymbol{u}_1+\frac{7}{11}\boldsymbol{u}_2+\boldsymbol{u}_3\end{cases},\text{即 } A=[\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3]=[\boldsymbol{u}_1\ \boldsymbol{u}_2\ \boldsymbol{u}_3]\begin{bmatrix}1&\frac{-1}{2}&\frac{1}{2}\\0&1&\frac{7}{11}\\0&0&1\end{bmatrix},$$

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$$\therefore A=\left[\frac{\boldsymbol{u}_1}{\|\boldsymbol{u}_1\|}\ \frac{\boldsymbol{u}_2}{\|\boldsymbol{u}_2\|}\ \frac{\boldsymbol{u}_3}{\|\boldsymbol{u}_3\|}\right]\begin{bmatrix}\|\boldsymbol{u}_1\|&\frac{-1}{2}\cdot\|\boldsymbol{u}_1\|&\frac{1}{2}\cdot\|\boldsymbol{u}_1\|\\0&1\cdot\|\boldsymbol{u}_2\|&\frac{7}{11}\cdot\|\boldsymbol{u}_2\|\\0&0&1\cdot\|\boldsymbol{u}_3\|\end{bmatrix}=\underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{-1}{\sqrt{22}}&\frac{3}{\sqrt{44}}\\0&\frac{4}{\sqrt{22}}&\frac{-1}{\sqrt{44}}\\0&\frac{2}{\sqrt{22}}&\frac{5}{\sqrt{44}}\\\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{22}}&\frac{-3}{\sqrt{44}}\end{bmatrix}}_{Q}\underbrace{\begin{bmatrix}\sqrt{2}&\frac{-\sqrt{2}}{2}&\frac{\sqrt{2}}{2}\\0&\frac{\sqrt{11}}{2}&\frac{7\sqrt{11}}{22}\\0&0&\frac{\sqrt{396}}{11}\end{bmatrix}}_{R}.$$

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20. Diagonalize matrix $A=\begin{bmatrix}1&1&5\\1&5&1\\5&1&1\end{bmatrix}$; that is to find matrices $\boldsymbol{P}$ and $\boldsymbol{D}$ such that $\boldsymbol{A}=\boldsymbol{PDP}^T$. Which values are **not** in $\boldsymbol{P}$ or $\boldsymbol{D}$ matrices ?

A. $1/\sqrt{3}$. B. $-1/\sqrt{8}$. C. $2/\sqrt{8}$. D. -1/2. E. 1/3.

20.【解】選(b), (d), (e).

解 $char_A(x)=\det(A-xI)=...=(7-x)(4-x)(-4-x)$, 得特徵根 : $7, 4, -4$,

考慮各特徵根所對應的特徵空間:

$$V(7)=\ker(A-7I)=\ker(\begin{bmatrix}-6&1&5\\1&-2&1\\5&1&-6\end{bmatrix})=\ker(\begin{bmatrix}0&-11&11\\1&-2&1\\0&11&-11\end{bmatrix})=span(\{\underbrace{\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}}_{v_1}\}),$$

$$V(4)=\ker(A-4I)=\ker(\begin{bmatrix}-3&1&5\\1&1&1\\5&1&-3\end{bmatrix})=\ker(\begin{bmatrix}0&4&8\\1&1&1\\0&-4&-8\end{bmatrix})=span(\{\underbrace{\frac{1}{\sqrt{6}}\begin{bmatrix}-1\\2\\-1\end{bmatrix}}_{v_2}\}),$$

$$V(-4)=\ker(A+4I)=\ker(\begin{bmatrix}5&1&5\\1&9&1\\5&1&5\end{bmatrix})=\ker(\begin{bmatrix}0&-44&0\\1&9&1\\0&-44&0\end{bmatrix})=span(\{\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}}_{v_3}\}),$$

取 $P=\begin{bmatrix}\frac{1}{\sqrt{3}}&\frac{-1}{\sqrt{6}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{3}}&\frac{2}{\sqrt{6}}&0\\\frac{1}{\sqrt{3}}&\frac{-1}{\sqrt{6}}&\frac{-1}{\sqrt{2}}\end{bmatrix}$, 可得 $D=\begin{bmatrix}7&0&0\\0&4&0\\0&0&-4\end{bmatrix}$.

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6. (10%) Factor the matrix $\mathbf{B}$ into $\mathbf{B} = \mathbf{L}\mathbf{D}\mathbf{L}^{\mathrm{T}}$.

$$\mathbf{B} = \begin{bmatrix} 5 & 5 & 0 \\ 5 & 7 & 2 \\ 0 & 2 & 9 \end{bmatrix}.$$

$$R_{12}^{-1} R_{23}^{-1} A = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 5 & 0 \\ 5 & 7 & 2 \\ 0 & 2 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 5 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

$$A = R_{23}^{1} R_{12}^{1} U = LU$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$A = LDL^T$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

more

$$A = L\sqrt{D}\sqrt{D}L^T$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{7} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{7} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = LL^T$$

$$A = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ \sqrt{5} & \sqrt{2} & 0 \\ 0 & \sqrt{2} & \sqrt{7} \end{bmatrix} \begin{bmatrix} \sqrt{5} & \sqrt{5} & 0 \\ 0 & \sqrt{2} & \sqrt{2} \\ 0 & 0 & \sqrt{7} \end{bmatrix}$$

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4. Consider the following matrix

$$A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$$

a) (5%) Find the eigenvalues and eigenvectors of $A^T A$.

b) (7%) Compute the Singular Value Decomposition of A.

c) (3%) Compute the rank 1 approximation of A.

(a)

$$A^T A = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix},$$

$char_{A^T A}(x) = \det(A^T A - xI) = \det(\begin{bmatrix} 8-x & -4 \\ -4 & 2-x \end{bmatrix}) = x(x-10)$, 得特徵根 10, 0.

$$V(10) = \ker(A^T A - 10I) = \ker\begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} = span\{\begin{bmatrix} 2 \\ -1 \end{bmatrix}\}.$$

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$$V(0) = \ker(A^T A - 0I) = \ker\begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} = span\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}.$$

$\therefore$ 特徵根 10, 對應的特徵向量是 $\begin{bmatrix} 2t \\ -t \end{bmatrix}, t \neq 0;$

特徵根 0, 對應的特徵向量是 $\begin{bmatrix} t \\ 2t \end{bmatrix}, t \neq 0.$

(b)

由 (a), 故得 A 的奇異值 $\sigma_1=\sqrt{10}, \sigma_2=0, \Sigma=\begin{bmatrix}\sqrt{10} & 0\\ 0 & 0\end{bmatrix}$, 且取 $V=\begin{bmatrix}\frac{2}{\sqrt5} & \frac{1}{\sqrt5}\\ \frac{-1}{\sqrt5} & \frac{2}{\sqrt5}\end{bmatrix}=[\boldsymbol{v}_1\ \ \boldsymbol{v}_2]$,

另外, 取 $\boldsymbol{u}_1=\frac{1}{\sigma_1}A\boldsymbol{v}_1=\frac{1}{\sqrt{10}}\begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt5}\\ \frac{-1}{\sqrt5}\end{bmatrix}=\frac{1}{\sqrt2}\begin{bmatrix}1\\ -1\end{bmatrix}$,

$\ker(A^T)=\ker(\begin{bmatrix}2 & -2\\ -1 & 1\end{bmatrix})=span\{\begin{bmatrix}1\\ 1\end{bmatrix}\}$, $\therefore$ 取 $\boldsymbol{u}_2=\frac{1}{\sqrt2}\begin{bmatrix}1\\ 1\end{bmatrix}$, $U=[\boldsymbol{u}_1\ \ \boldsymbol{u}_2]=\begin{bmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\\ \frac{-1}{\sqrt2} & \frac{1}{\sqrt2}\end{bmatrix}$,

則 $A=U\Sigma V^T=\begin{bmatrix}\frac{1}{\sqrt2} & \frac{1}{\sqrt2}\\ \frac{-1}{\sqrt2} & \frac{1}{\sqrt2}\end{bmatrix}\begin{bmatrix}\sqrt{10} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt5} & \frac{1}{\sqrt5}\\ \frac{-1}{\sqrt5} & \frac{2}{\sqrt5}\end{bmatrix}^T$.

(c)

A 的 $rank\ 1-$近似為 $\sigma_1\boldsymbol{u}_1\boldsymbol{v}_1^T=\sqrt{10}\begin{bmatrix}\frac{1}{\sqrt2}\\ \frac{-1}{\sqrt2}\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt5} & \frac{-1}{\sqrt5}\end{bmatrix}=\begin{bmatrix}2 & -1\\ -2 & 1\end{bmatrix}$.

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20. Find a singular value decomposition $\boldsymbol{A}=\boldsymbol{U\Sigma V}^{\mathrm{T}}$ with $\boldsymbol{U}$ and $\boldsymbol{V}$ being both orthogonal matrices, where $\boldsymbol{A}=\begin{bmatrix}3 & 2 & 2\\ 2 & 3 & -2\end{bmatrix}$. Which values are **not** in $\boldsymbol{U}$ or $\boldsymbol{V}$ matrices?

(a) $1/\sqrt2$ (b) $1/\sqrt3$. (c) 1/2. (d) 1/3. (e) $1/(3\sqrt2)$.

注意:背面有試題

10.【解】(b), (c). 【98 中央資工類題】

$AA^T=\begin{bmatrix}17 & 8\\ 8 & 17\end{bmatrix}$ 令為 B, $char_B(x)=x^2-34x+225$, 得特徵根25, 9.

故得 A 的奇異值 $\sigma_1=5, \sigma_2=3$, $\therefore\Sigma=\begin{bmatrix}5 & 0 & 0\\ 0 & 3 & 0\end{bmatrix}$,

$V(25)=\ker(B-25I)=\ker(\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix})=span\{\begin{bmatrix}1\\1\end{bmatrix}\}$;

$V(9)=\ker(B-9I)=\ker(\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix})=span\{\begin{bmatrix}1\\-1\end{bmatrix}\}$,

故取$U=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}=[\boldsymbol{u}_1\ \ \boldsymbol{u}_2]$,

又 $\boldsymbol{v}_1=\frac{1}{\sigma_1}A^T\boldsymbol{u}_1=\frac{1}{5}\begin{bmatrix}3 & 2\\ 2 & 3\\ 2 & -2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ 0\end{bmatrix}$, 取$\boldsymbol{v}_2\frac{1}{\sigma_2}A^T\boldsymbol{u}_2=\frac{1}{3}\begin{bmatrix}3 & 2\\ 2 & 3\\ 2 & -2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{bmatrix}=\begin{bmatrix}\frac{1}{3\sqrt{2}}\\ \frac{-1}{3\sqrt{2}}\\ \frac{4}{3\sqrt{2}}\end{bmatrix}$,

再考慮$N(A)$, $N(A)=\ker(\begin{bmatrix}3 & 2 & 2\\ 2 & 3 & -2\end{bmatrix})=span\{\begin{bmatrix}-2\\2\\1\end{bmatrix}\}$,

得$V=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3}\\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3}\\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3}\end{bmatrix}$, 且 $A=\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}5 & 0 & 0\\ 0 & 3 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{3\sqrt{2}} & \frac{-2}{3}\\ \frac{1}{\sqrt{2}} & \frac{-1}{3\sqrt{2}} & \frac{2}{3}\\ 0 & \frac{4}{3\sqrt{2}} & \frac{1}{3}\end{bmatrix}^T=U\sum V^T$.

105 師大

5. (10 分) Let $M=\begin{bmatrix}3 & 0 & 4\\ 5 & 0 & 0\end{bmatrix}$. (a) Determine whether M^TM is diagonalizable or not. If M^TM is diagonalizable, find a diagonalization for it. (b) Find the singular value decomposition of M.

$$(a)M^TM=\begin{bmatrix}3&5\\0&0\\4&0\end{bmatrix}\begin{bmatrix}3&0&4\\5&0&0\end{bmatrix}=\begin{bmatrix}34&0&12\\0&0&0\\12&0&16\end{bmatrix}$$(令為 A) 為實對稱矩陣, 故可對角化.

$$char_A(x)=\det(A-xI)=\det(\begin{bmatrix}34-x&0&12\\0&0&0\\12&0&16-x\end{bmatrix})=(0-x)[(34-x)(16-x)-144]$$

$=(0-x)(x^2-50x+400)=(0-x)(10-x)(40-x)$, $\therefore$ 特徵根為 0, 10, 40,

0 的特徵空間 $V(0)=\ker(A-0I)=\ker(\begin{bmatrix}34&0&12\\0&0&0\\12&0&16\end{bmatrix})=span\{\begin{bmatrix}0\\1\\0\end{bmatrix}\}$,

10 的特徵空間 $V(10)=\ker(A-10I)=\ker(\begin{bmatrix}24&0&12\\0&-10&0\\12&0&6\end{bmatrix})=span\{\begin{bmatrix}1\\0\\-2\end{bmatrix}\}$,

40 的特徵空間 $V(40)=\ker(A-40I)=\ker(\begin{bmatrix}-6&0&12\\0&-40&0\\12&0&-24\end{bmatrix})=span\{\begin{bmatrix}2\\0\\1\end{bmatrix}\}$,

故取 $P=\begin{bmatrix}0&1&2\\1&0&0\\0&-2&1\end{bmatrix}$ 可得 $P^{-1}AP=D=\begin{bmatrix}0&0&0\\0&10&0\\0&0&40\end{bmatrix}$.

(b)

由 (a) 得 M^TM 特徵根 40, 10, 0.

故得 M 的奇異值 $\sigma_1=\sqrt{40},\sigma_2=\sqrt{10}$, $\therefore \Sigma=\begin{bmatrix}\sqrt{40}&0&0\\0&\sqrt{10}&0\end{bmatrix}$,

且 M^TM 的特徵向量單範化得 $V=\begin{bmatrix}\frac{2}{\sqrt5}&\frac{1}{\sqrt5}&0\\0&0&1\\\frac{1}{\sqrt5}&\frac{-2}{\sqrt5}&0\end{bmatrix}=[\boldsymbol{v}_1\ \ \boldsymbol{v}_2\ \boldsymbol{v}_3]$,

取 $\frac{1}{\sqrt{40}}M\boldsymbol{v}_1=\frac{1}{\sqrt{40}}\begin{bmatrix}\frac{10}{\sqrt5}\\\frac{10}{\sqrt5}\end{bmatrix},\frac{1}{\sqrt{10}}M\boldsymbol{v}_2=\frac{1}{\sqrt{10}}\begin{bmatrix}\frac{-5}{\sqrt5}\\\frac{5}{\sqrt5}\end{bmatrix}$, 單範化後得 $\boldsymbol{u}_1=\begin{bmatrix}\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}\end{bmatrix},\boldsymbol{u}_2=\begin{bmatrix}\frac{-1}{\sqrt2}\\\frac{1}{\sqrt2}\end{bmatrix}$,

$\therefore$ 取 $U=\begin{bmatrix}\frac{1}{\sqrt2}&\frac{-1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}$, 可得 $M=U\Sigma V^T$, 為M的奇異值分解.

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5. (50%)Give the matrix $\mathbf{M} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Solve the following questions.

(a) (10%) Describe the nullspace of $\mathbf{M}$. (The nullspace of $\mathbf{M}$ consists of all solutions to $\mathbf{Mx} = 0$.)

(b) (5%) Compute $\mathbf{M}^T\mathbf{M}$ and $\mathbf{M}\mathbf{M}^T$.

(c) (5%) Explain whether $\mathbf{M}^T\mathbf{M}$ is similar to $\mathbf{M}\mathbf{M}^T$.

(d) (10%) Compute the eigenvalues and eigenvectors of $\mathbf{M}^T\mathbf{M}$.

(e) (10%) Diagonalize $\mathbf{M}^T\mathbf{M}$.

(f) (10%) Find the singular value decomposition of $\mathbf{M}$.

(a) Nullspace of M

The nullspace $N(M)$ is the set of all vectors $\mathbf{x}$ such that $M\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 = 0 \implies x_1 = -x_2$

$x_2 + x_3 = 0 \implies x_3 = -x_2$

Let $x_2 = c$ (free variable). Then $\mathbf{x} = \begin{bmatrix} -c \\ c \\ -c \end{bmatrix} = c \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

The nullspace is the span of the vector $\begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$.

(b) Compute $M^T M$ and MM^T

$$M^T = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$M^T M$ (3×3):

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

MM^T (2×2):

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(c) Is M^TM similar to MM^T?

No. Similar matrices must have the same dimensions. M^TM is a 3×3 matrix, while MM^T is a 2×2 matrix. Therefore, they cannot be similar. However, they do share the same non-zero eigenvalues.

(d) Eigenvalues and Eigenvectors of M^TM

To find the eigenvalues λ, we solve $\det(M^TM - \lambda I) = 0$:

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

Expanding along the first row:$(1-\lambda)[(2-\lambda)(1-\lambda) - 1] - 1(1-\lambda) = 0(1-\lambda)[\lambda^2 - 3\lambda + 1] - (1-\lambda) = 0(1-\lambda)(\lambda^2 - 3\lambda) = 0 \implies -\lambda(\lambda-1)(\lambda-3) = 0$

Eigenvalues: $\lambda_1 = 3, \lambda_2 = 1, \lambda_3 = 0.$

Eigenvectors:

For $\lambda_1 = 3$: $\mathbf{v}_1 = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$ (normalized: $\frac{1}{\sqrt{6}} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}^T$)

For $\lambda_2 = 1$: $\mathbf{v}_2 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$ (normalized: $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$)

For $\lambda_3 = 0$: $\mathbf{v}_3 = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$ (normalized: $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T$)

(e) Diagonalize M^TM

Since M^TM is symmetric, it is diagonalized by $M^TM = V\Lambda V^T$, where V is the orthogonal matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues.

$$V = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(f) Singular Value Decomposition (SVD) of M

The SVD is $A_{2\times3} = U_{2\times2}\Sigma_{2\times3}V^T_{3\times3}$.

$$U = [\underbrace{\mathbf{u_1} \ldots \mathbf{u_r}}_{R(A)} | \underbrace{\mathbf{u_{r+1}} \ldots \mathbf{u_m}}_{N(A^T)}], V = [\underbrace{\mathbf{v_1} \ldots \mathbf{v_r}}_{R(A^T)} | \underbrace{\mathbf{v_{r+1}} \ldots \mathbf{v_n}}_{N(A)}]$$

Σ: Square roots of non-zero eigenvalues of M^TM.$\sigma_1 = \sqrt{3}, \sigma_2 = \sqrt{1}$. Thus, $\Sigma = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

V: The eigenvectors of M^TM found in part (d).

$$v_3 = N(A) = ker \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

U: Found via $u_i = \frac{1}{\sigma_i}Mv_i$.

$$u_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$u_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

Result: $M = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}^T$

108 交大

3. Let $C[-1,1]$ be the vector space over R of all continuous functions defined on the interval $[-1,1]$. Let $V:\{f(x)\in C[-1,1] | f(x)=ae^x+be^{2x}+ce^{3x}, a,b,c\in R\}$.

a. **(2 points)** Prove that V is a subspace of $C[-1,1]$.

b. **(5 points)** Prove that $B=\{e^x,e^{2x},e^{3x}\}$ is a basis of V.

c. **(5 points)** Prove that $B'=\{e^x-2e^{3x}, e^x+e^{2x}+2e^{3x}, 3e^{2x}+e^{3x}\}$ is a basis of V.

108-交大資工

(a)

零函數 $=0e^x+0e^{2x}+0e^{3x}\in V, \therefore V\neq\varnothing,$

任取 $f(x)=ae^x+be^{2x}+ce^{3x}\in V,\ g(x)=le^x+me^{2x}+ne^{3x}\in V, \alpha\in R,$

則 $\alpha f(x)+g(x)=\alpha(ae^x+be^{2x}+ce^{3x})+(le^x+me^{2x}+ne^{3x})$

$$=(\alpha a+l)e^x+(\alpha b+m)e^{2x}+(\alpha c+n)e^{3x}\in V,$$

即滿足向量加法與純量積封閉性,故 V 為 $C[-1,1]$ 的一個子空間.

(b)

$$Wronskian[e^x,e^{2x},e^{3x}]=\det(\begin{bmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x}\end{bmatrix})=\ldots=2e^{6x}\neq 0, \forall x\in R,$$

所以 B 是線性獨立的

又 B 生成 V, 故 B 為 V 的一個基底

(c)

設 $\alpha(e^x-2e^{3x})+\beta(e^x+e^{2x}+2e^{3x})+\gamma(3e^{2x}+e^{3x})=0,$

則 $(\alpha+\beta)e^x+(\beta+3\gamma)e^{2x}+(-2\alpha+2\beta+\gamma)e^{3x}=0,$

但 $\{e^x,e^{2x},e^{3x}\}$ 是線性獨立集, $\therefore \begin{cases} \alpha+\beta=0 \\ \beta+3\gamma=0 \\ -2\alpha+2\beta+\gamma=0 \end{cases}$

可解得 $\therefore \alpha=\beta=\gamma=0.$

$\therefore B'=\{e^x-2e^{3x}, e^x+e^{2x}+2e^{3x}, 3e^{2x}+e^{3x}\}$ 是線性獨立集,

又由 (b) 可得 $\dim(V)=3$, 故 B' 可為 V 的一基底.

106 師大

3. (10 分) Determine an explicit description of $T(\boldsymbol{x})$ using the given basis $\mathcal{B}$ and the matrix representation of T respective to $\mathcal{B}$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \text{ and } \mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(Hint: *It needs to find the standard matrix of T.*)

3.【解】

令 $S=\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ 為 R^3 上的標準基底, 則 B 到 S 的轉換矩陣 $[I_{R^3}]_B^S = P = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$,

$$\therefore [T]_S = [I_{R^3}]_B^S [T]_B [I_{R^3}]_S^B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ -1 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 6 & -5 \\ -4 & 4 & +5 \\ -1 & 3 & 1 \end{bmatrix},$$

$$\therefore T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+6y-5z \\ -4x+4y+5z \\ -x+3y+z \end{bmatrix}.$$

110 台科

4. (10%, multiple choice question, no proof needed, points will only be awarded if all correct answers are selected) The singular value decomposition (SVD) can be leveraged to factorize a real value matrix A into a product of three simpler matrices $A = U\Sigma V^T$, where T denotes transpose operator. Which of the following statements are correct about the SVD decomposition?
   (a) $UU^T = I$
   (b) $U^T U \neq I$
   (c) $VV^T \neq I$
   (d) $V^T V = I$
   (e) $V^T A^T = \Sigma U^T$
   (f) $\Sigma V^T = AU^T$
   (g) The result of performing SVD for a given matrix is unique
   (h) $\text{rank}(A) = \text{rank}(\Sigma)$
   (i) The column vectors of V are eigenvectors of $A^T A$
   (j) The row vectors of U are eigenvectors of AA^T

<sol> A,D,H,I

108 中正

4. (7%) Find a 3×3 symmetric matrix whose eigenvalues are $\lambda_1 = 4$, $\lambda_2 = 2$, $\lambda_3 = 0$ and for which the corresponding eigenvectors are $\mathbf{v}_1 = (1, 1, 0)^T$, $\mathbf{v}_2 = (0, 0, 1)^T$, $\mathbf{v}_3 = (-1, 1, 0)^T$.

4.【解】

因$\{v_1, v_2, v_3\}$為正交集,故可取 $A = 4\cdot\frac{1}{2}\cdot v_1v_1^T + 2\cdot v_2v_2^T + 0\cdot\frac{1}{2}\cdot v_3v_3^T$

$$= 2\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ 滿足所求.

Spectral Decomposition/Eigen-decomposition

當矩陣 A 滿足：

- 是 **實對稱矩陣**（$A = A^T$），或
- 是 **Hermitian 矩陣**（複數情況）

就一定可以寫成：

$$A = \sum_i \lambda_i v_i v_i^T$$

其中：

- λ_i：特徵值
- v_i：**正交且單位長度**的特徵向量
- $v_i v_i^T$：投影矩陣

若 A 是 **對稱矩陣（或可正交對角化）**，則：

$$\boxed{A = \lambda_1\, vv^T + \lambda_2\, uu^T + \lambda_3\, ss^T}$$

條件是：

- v, u, s 是 **單位特徵向量**
  $\|v\| = \|u\| = \|s\| = 1$
- 彼此正交
- $\lambda_1, \lambda_2, \lambda_3$ 為對應的 eigenvalues

如果 v, u, s **不是單位向量**

那正確寫法應該是：

$$\boxed{A = \lambda_1 \frac{vv^T}{v^T v} + \lambda_2 \frac{uu^T}{u^T u} + \lambda_3 \frac{ss^T}{s^T s}}$$

110 師大

3. (12 points) Find an explicit formula for the reflection operator T_w of $\mathcal{R}^3$ about the plane W with equation $x - 2y + 3z = 0$.

1. 核心概念

給定平面的單位法向量 $\mathbf{u}$，反射算子 T_W 可以表示為：

$$T_W(\mathbf{x}) = (I - 2\mathbf{u}\mathbf{u}^T)\mathbf{x}$$

其中：

I 是單位矩陣。

$\mathbf{u}$ 是平面的 單位法向量（必須將長度正規化為 1）。

$\mathbf{u}\mathbf{u}^T$ 是投影矩陣，將向量投影到法向量的方向上。

2. 具體推導步驟

第一步：找出法向量並正規化

平面的法向量為 $\mathbf{n} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$。其長度為 $\|\mathbf{n}\| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14}$。

因此，單位法向量 $\mathbf{u}$ 為：

$$\mathbf{u} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

第二步：計算 $\mathbf{u}\mathbf{u}^T$

$$\mathbf{u}\mathbf{u}^T = \frac{1}{14} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}$$

第三步：計算反射矩陣 $H = I - 2\mathbf{u}\mathbf{u}^T$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{14} \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}$$

$$H = \frac{1}{7} \left(\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix} \right) = \frac{1}{7} \begin{bmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ -3 & 6 & -2 \end{bmatrix}$$

3. 最後答案 (Explicit Formula)

將矩陣與向量 $\begin{bmatrix} x & y & z \end{bmatrix}^T$ 相乘，得到：

$$T_W \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{6x+2y-3z}{7} \\ \frac{2x+3y+6z}{7} \\ \frac{-3x+6y-2z}{7} \end{bmatrix}$$

110 師大-2

5. (12 points) Find an orthogonal operator T on $\mathcal{R}^3$ such that $T(\boldsymbol{v_1}) = \boldsymbol{w_1}$ and $T(\boldsymbol{v_2}) = \boldsymbol{w_2}$, where $\boldsymbol{v_1} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\boldsymbol{v_2} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $\boldsymbol{w_1} = \frac{1}{7}\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ and $\boldsymbol{w_2} = \frac{1}{7}\begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}$.

已知：

$$v_1 = \frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, v_2 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$w_1 = \frac{1}{7} \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}, w_2 = \frac{1}{7} \begin{bmatrix} 6 \\ 2 \\ -3 \end{bmatrix}$$

第一步：驗證長度與正交性

一個算子要是正交的，前提是映射前後的向量性質必須一致：

長度 (Norms):

$\|v_1\| = \frac{1}{3}\sqrt{1^2+2^2+2^2} = 1$, $\|w_1\| = \frac{1}{7}\sqrt{2^2+3^2+6^2} = 1$ (符合)

$\|v_2\| = \frac{1}{3}\sqrt{2^2+1^2+(-2)^2} = 1$, $\|w_2\| = \frac{1}{7}\sqrt{6^2+2^2+(-3)^2} = 1$ (符合)

夾角 (Dot Product):

$v_1 \cdot v_2 = \frac{1}{9}(1 \cdot 2 + 2 \cdot 1 + 2 \cdot (-2)) = 0$

$w_1 \cdot w_2 = \frac{1}{49}(2 \cdot 6 + 3 \cdot 2 + 6 \cdot (-3)) = 0$

兩組向量皆為單位正交向量組 (Orthonormal sets)，這表示該正交算子 T 確實存在。

第二步：找出第三個基底向量

為了建立完整的正交矩陣 Q，我們需要利用外積 (Cross Product) 找出與前兩者垂直的第三個單位向量。

計算 $v_3 = v_1 \times v_2$:

$$v_3 = \frac{1}{9}\begin{vmatrix} i & j & k \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{vmatrix} = \frac{1}{9}\begin{bmatrix} -6 \\ 6 \\ -3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

計算 $w_3 = w_1 \times w_2$:

$$w_3 = \frac{1}{49}\begin{vmatrix} i & j & k \\ 2 & 3 & 6 \\ 6 & 2 & -3 \end{vmatrix} = \frac{1}{49}\begin{bmatrix} -21 \\ 42 \\ -14 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -3 \\ 6 \\ -2 \end{bmatrix}$$

第三步：求正交矩陣 T

我們知道正交矩陣 T 滿足 $TV = W$，其中 $V = [v_1, v_2, v_3]$ 且 $W = [w_1, w_2, w_3]$。

由於 V 與 W 都是正交矩陣，其反矩陣等於轉置矩陣（即 $V^{-1} = V^T$）。

$$T = WV^T = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \begin{bmatrix} v_1^T \\ v_2^T \\ v_3^T \end{bmatrix}$$

矩陣相乘計算：

$$T = \frac{1}{7}\begin{bmatrix} 2 & 6 & -3 \\ 3 & 2 & 6 \\ 6 & -3 & -2 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$$

$$T = \frac{1}{21}\begin{bmatrix} (2+12+6) & (4+6-6) & (4-12+3) \\ (3+4-12) & (6+2+12) & (6-4-6) \\ (6-6+4) & (12-3-4) & (12+6+2) \end{bmatrix}$$

最終答案：

$$T = \frac{1}{21}\begin{bmatrix} 20 & 4 & -5 \\ -5 & 20 & -4 \\ 4 & 5 & 20 \end{bmatrix}$$

這個矩陣 T 即為所求的正交算子，它將 v_1, v_2 映射至 w_1, w_2。